Shannon theorem revisited…

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Abstract

We state (after Claude Shannon) that if $x(t)$ is bandlimited, *i.e.* its Fourier transform is compactly supported in $[-W, W]$, then

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} x(nT) \cdot \frac{\sin\left(\frac{\pi}{T}(t-nT)\right)}{\frac{\pi}{T}(t-nT)} \\ &= \sum_{n=-\infty}^{\infty} x(nT) \cdot \mathrm{sinc}\left(\frac{\pi}{T}(t-nT)\right), \end{aligned}$$

where $T = 2W$.

Proof: For $x(t)$, its Fourier **transform** is given as

$$\mathcal{X}(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi \cdot t\omega} dt.$$

Its inverse yields $x(t)$ again

$$x(t) = \int_{-\infty}^{\infty} \mathcal{X}(\omega)\, e^{+j2\pi\cdot\omega t} d\omega = \int_{-W}^{W} [\mathcal{X}(\omega)] \cdot e^{+j2\pi\cdot\omega t} d\omega.$$

Since $\mathcal{X}(\omega)$ is compactly supported, we can expand it into a Fouries **series**

$$\mathcal{X}(\omega) = \sum_{n=-\infty}^{\infty} \alpha_n \cdot e^{-j2\pi\cdot\frac{n\omega}{2W}},$$

where

$$\alpha_n = \frac{1}{2W}\int_{-W}^{W} \mathcal{X}(\omega) \cdot e^{+j2\pi\cdot\frac{n\omega}{2W}} d\omega = \frac{1}{2W} \cdot x\left(\frac{n}{2W}\right) \text{ (why!?).}$$

Thus

$$\mathcal{X}(\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{2W} x\left(\frac{n}{2W}\right) \cdot e^{-j2\pi\cdot\frac{n\omega}{2W}}$$

and

$$\begin{aligned} x(t) &= \int_{-W}^{W} \left[\sum_{n=-\infty}^{\infty} \frac{1}{2W} x\left(\frac{n}{2W}\right) \cdot e^{-j2\pi\cdot\frac{n\omega}{2W}}\right] e^{+j2\pi\cdot\omega t} d\omega \\ &= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \cdot \underbrace{\frac{1}{2W}\int_{-W}^{W} e^{+j2\pi\cdot\omega\left(t-\frac{n}{2W}\right)} d\omega}_{=\mathrm{sinc}\left(\frac{\pi}{T}(t-nT)\right) \text{ (why?!)}}. \end{aligned}$$

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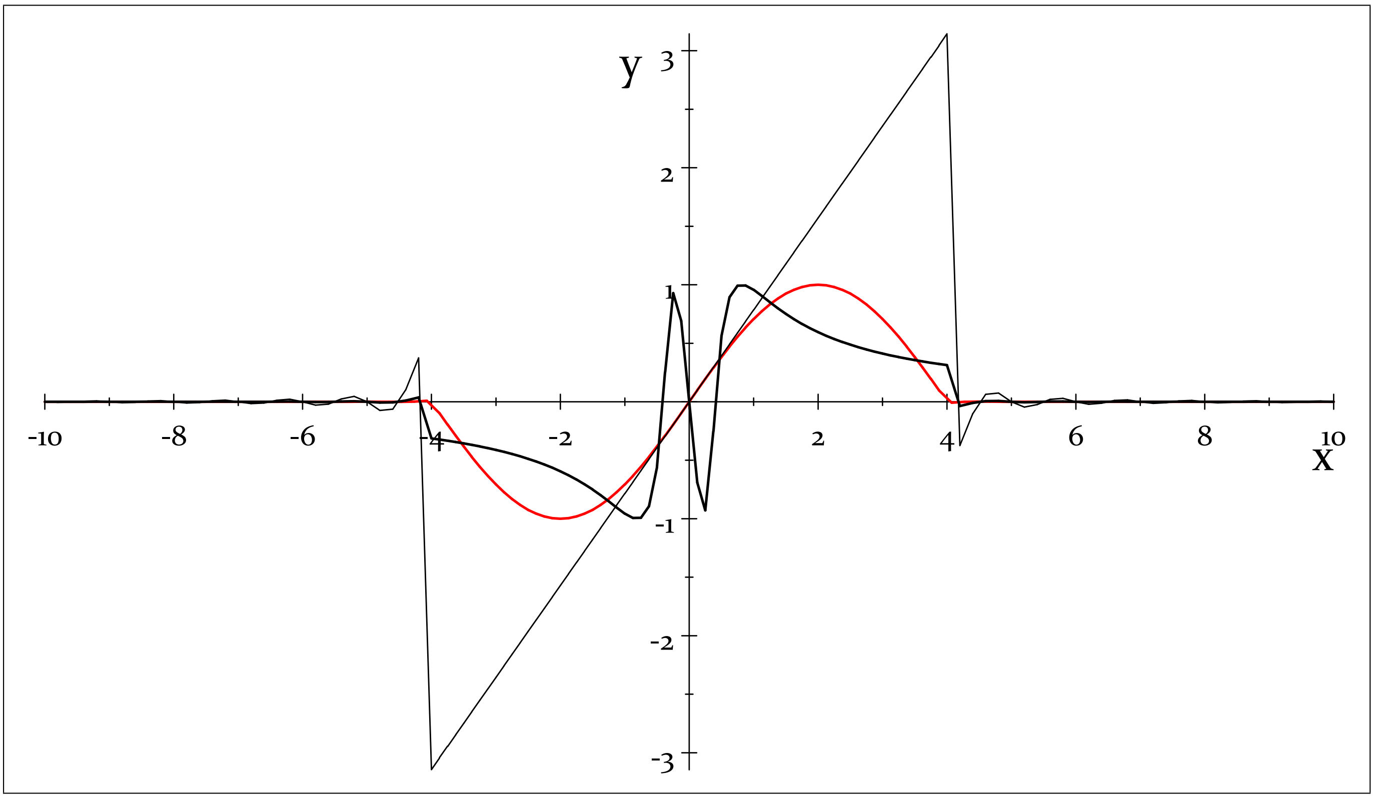

The interpolation formulas are as simple as that

$$
\begin{aligned}
f(x,T,N) &= \sum_{n=-N}^{N} \sin\left(\frac{n\pi}{N}\right) \cdot s(x,T,n) \\
l(x,T,N) &= \sum_{n=-N}^{N} \frac{n\pi}{N} \cdot s(x,T,n) \\
h(x,T,N) &= \sum_{n=-N}^{-1} \sin\left(\frac{N}{n\pi}\right) \cdot s(x,T,n) + \sum_{n=1}^{N} \sin\left(\frac{N}{n\pi}\right) \cdot s(x,T,n)
\end{aligned}
$$

Where we have a $\operatorname{sinc}(x)$ function defined as

$$s(x,m,n) = \frac{\sin(\pi(mx-n))}{\pi(mx-n)}.$$

Guess the original function which interpolation has such a Batman-like shape. . .